Introduction to Fractals and Holomorphic Dynamics

for Scientific Computing in Mathematica

Contents

Abstract

Fractals are beautiful shapes that achieve to reach nearly everyone in the scientific world at least once in their career. In a way are they complex and simple at the same time and fascinate not only mathematicians but also artists and other scientists. In this article I want to introduce you to the exciting world of fractals and holomorphic dynamics. I will cover the necessary mathematics like definitions and examples at a beginner or rather freshman level, and get practical by covering a few of the most important and well known fractals. I will not explain things like complex numbers or how exactly Newton's method works though. There are many great resources on those topics online. All the visuals are made using Wolfram Mathematica 13. This article was written in the context of the university course "Scientific Computing in Mathematica" at the University of Potsdam.

1 Introduction

In a way are fractals an antipole towards calculus. While in calculus things are assumed to get smoother and smoother the closer you get, fractal geometry to some extent celebrates the roughness in nature. Fractals are a beautiful blend between simplicity and complexity and the mathematically constructed ones often include infinitely repeating patterns. In particular, computer scientists like me find them very fascinating given the fact that it takes so little code to create images of fractals that are way more intricate than anything humans could ever create with pen and paper.

I feel like nearly everyone in the scientific world has heard of the Mandelbrot set and seen a picture of it at least once in their life. It is magical and fascinates people, but only a few actually know the correct definition of a fractal. At least not to the one Benoît Mandelbrot the father of fractal geometry, had in mind. Mandelbrot was the one who coined the term "fractal" in 1975 and wrote about them in depth in his book "Les Objets Fractals: Forme, Hasard et Dimension" ("Fractals: Form, Chance and Dimension") [3]. Mandelbrot based the term on the Latin "frāctus", meaning "broken" or "fractured". This also shows just how young the fractal geometry still is. While Mandelbrot truly was a visionary and used the technology of his time to make great scientific progress, many other mathematicians also deserve credit for laying the foundations for his work in the centuries preceding him. To get the full picture, it helps to get familiar with the relevant work of Karl Weierstrass, Georg Cantor, Felix Hausdroff, Gastion Julia, Pierre Fatour and Paul Lévy [1]. Not only to make Mandelbrot's work clearer but to see its connections to other branches of mathematics as well. Some of those names will also reappear later in this article.

A common misconception is that fractals have to be perfectly self-similar, meaning that the shapes contain a smaller but exact replica of themselves like the Sierpinski triangle.
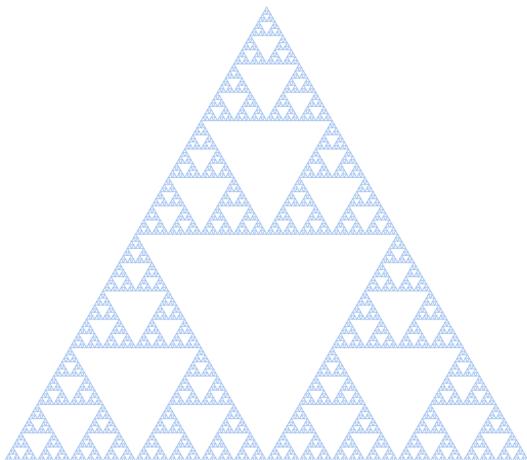

Figure 1: Sierpinski triangle (drawn to step 7).

But fractals go way beyond just self-similar fractals. They aren't even limited to geometric patterns, but can also describe processes in time [5]. Self-similar shapes give a basis for modeling the regularity in some forms of roughness. Mandelbrot had a much broader concept in mind though. One motivated not by beauty but more by a pragmatic desire to model nature in a way that captures roughness [16]. And in fact, have fractals a broad range of applications nowadays. For example, are many natural phenomena known to have fractal features which help study them. For instance, are the fractal properties of leaves currently being used to determine how much carbon is contained in trees. Additionally are there many applications in technology and fractal art is a big topic in the mathematical creative world [5]. Fractals are also particularly important to chaos theory with many overlapping research topics.

Figure 2: Romanesco broccoli showing fractal features [9].

In this short article, I want to give you an introduction to fractal geometry and a topic known as "Holomorphic Dynamics" which you will recognize as one of the essential pieces that shows up in many of the most important fractals. I will explain what fractals are, show how to identify fractals and go into detail and apply the theoretical basis to a few of the most well-known fractals. Namely, the Mandelbrot set and Julia sets, Newton's fractal and the bifurcation diagram of the logistic map. My work will build up to a fascinating connection between the bifurcation diagram of the logistic map and the Mandelbrot set to excite and inspire you to continue your own research. All the visuals are implemented in Wolfram Mathematica 13 and you can find the code at the end.

Now: What are fractals? How do I recognize them? How do you generate them? And how do I do all those things in Mathematica?

2 Fractals & Holomorphic Dynamics

2.1 Holomorphic Dynamics

As already announced I first want to introduce you to holomorphic dynamics.

The "Holomorphic" in "Holomorphic Dynamics" simply refers to functions that have a complex number input, a complex number output ($f : \mathbb{C} \to \mathbb{C}$) and which are also differentiable, which in this case (roughly speaking) means that going from input to output looks like scaling and rotating (see figure 3) [15] [8] [4]. This includes most of the basic functions. For example polynomial, exponential, trigonometric functions and any combination of them:

$$z^2, e^z, \sin(z), \ldots$$

The "Dynamics" on the other hand in this case refers to applying this function over and over creating a sequence of numbers and seeing what happens over many iterations. Which can be expressed as something like this:

$$z_{n+1} = f(z_n)$$

The part that is important to fractals is that dynamics like these can resolve (after many iterations) into a few different states. They can either result in a cycle of an arbitrary length with repeating values (e.g. $z_0 \to z_1 \to z_2 \to z_0$ a cycle of length 3), converge to a single point ($\lim_{i \to \infty} z_i = a \in \mathbb{C}$), diverge (converge to infinity) or show bound chaotic behavior that goes on forever.

Exactly when visualizing when these possible behaviors arise for a holomorphic function applied on different starting values you often get some kind of fractal pattern.

To make things clear let us consider the following example of a holomorphic function f as a dynamic system.

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = z \cdot e^{\frac{\pi}{2}i}, \quad z_{k+1} = z_k \cdot e^{\frac{\pi}{2}i}$$

For any input $z_0 \in \mathbb{C}$, creates f a four-cycle by rotating the complex number in the complex plane 90° counterclockwise four times and landing right where z_0 started.

$$z_0 \xrightarrow{f} z_1 \xrightarrow{f} z_2 \xrightarrow{f} z_3 \xrightarrow{f} z_0$$

This can easily be shown with Euler's identity $e^{\pi i} = -1$.

$$\begin{aligned}
f(z_0) = z_1 &= z_0 \cdot e^{\frac{\pi}{2}i} \\
f(z_1) = z_2 &= \left(z_0 \cdot e^{\frac{\pi}{2}i}\right) \cdot e^{\frac{\pi}{2}i} \\
f(z_2) = z_3 &= \left(\left(z_0 \cdot e^{\frac{\pi}{2}i}\right) \cdot e^{\frac{\pi}{2}i}\right) \cdot e^{\frac{\pi}{2}i} \\
f(z_3) = z_4 &= \left(\left(\left(z_0 \cdot e^{\frac{\pi}{2}i}\right) \cdot e^{\frac{\pi}{2}i}\right) \cdot e^{\frac{\pi}{2}i}\right) \cdot e^{\frac{\pi}{2}i} \\
&= z_0 \cdot e^{2\pi i} = z_0
\end{aligned}$$

For example with $z_0 = 1$:

$$f(1) = i \quad f(i) = -1 \quad f(-1) = -i \quad f(-i) = 1$$

On the other hand a simple divergent function for example is $f(z) = 4z$.
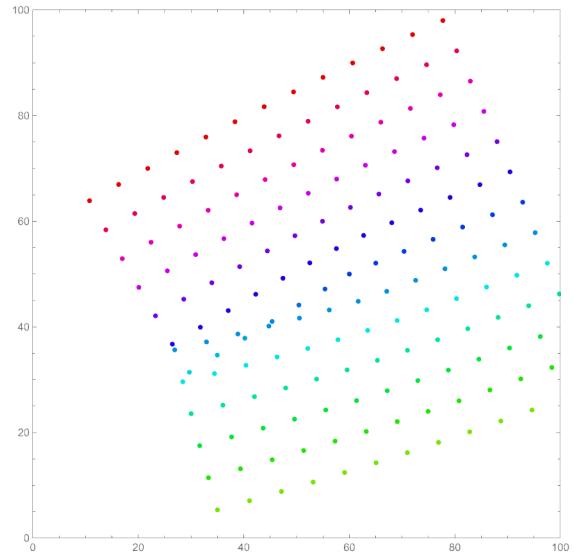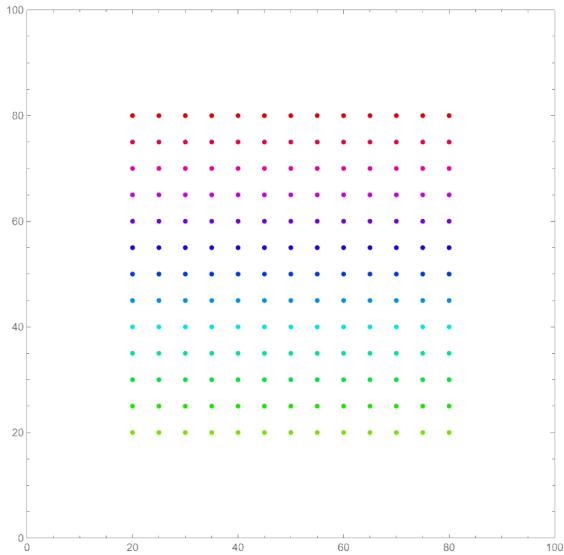
Figure 3: Example of a holomorphic function applied on a grid of points.

2.2 Fractal geometry

Fractal geometry is the branch of mathematics that originates fractals and deals with their definition and characteristics. It in itself lies in the mathematical branch of measure theory [5]. To build a fundamental understanding of what fractals are and to be able to really appreciate the fractals I will introduce to you later we first have to understand the proper definition of a fractal and how we can recognize them.

2.2.1 Fractal dimension

Fundamental to all fractals by underlying the definition itself is the so-called fractal dimension. The fractal dimension is a concept that lets us measure the roughness of a shape on different scales. More formally is it a ratio providing a statistical index of complexity comparing how details in a pattern change with the scale at which they are measured [6].

The fractal dimension can change based on how far you are "zoomed" in. Most definitions of a fractal dimension therefore just focus on what limit this dimension approaches for a given shape the closer we get. This only works for infinite fractals that we constructed ourselves tough and not for naturally occurring fractals. With natural fractals at some point, you will just hit the scale of atoms. For an applied setting in nature, it is therefore most common to just look at a wide range of scales and see if the dimension stays roughly constant. With different techniques, we can determine or measure a fractal dimension for nearly every shape. Some analytically and some numerically. There are many differently defined fractal dimensions.

To begin with, I want to introduce you to the fractal dimension with 4 perfectly self-similar shapes, three of which are not even fractals but will help us gain an intuition for what the fractal dimension is. The shapes are a line, a square, a cube and the already shown Sierpinski triangle.

These shapes are self-similar because a line can be broken up into two smaller lines, each of which is a perfect copy of the original just scaled down by half. A square can be broken down into four smaller squares scaled down by half. A cube can be broken down into eight smaller cubes again where each one is a scaled-down version of the original. And lastly, the Sierpinski triangle is made up of three exact copies of itself where the side length of the smaller copies is exactly one-half of the original length. To generalize the way we talk about measuring the length, area, volume or their 2.5-dimensional equivalent for example it is most intuitive to talk about mass instead. When scaling down the line by a half the mass also scales down by a half $\left(\frac{M}{2}\right)$. If we scale down the square by a half the mass scales down by a fourth $\left(\frac{M}{4} = M \cdot \left(\frac{1}{2}\right)^2\right)$ because it takes four copies of the smaller square to form the original one. Likewise if you scale down the cube by one half, the mass is scaled down by one eighth $\left(\frac{M}{8} = M \cdot \left(\frac{1}{2}\right)^3\right)$. If we now apply the same reasoning to the Sierpinski triangle it only makes sense that if we scale down the Sierpinski triangle by one-half its mass gets scaled down by one-third $\left(\frac{M}{3}\right)$ because it takes three copies of the scaled-down versions to form the original triangle.

What stands out is that for the line, square and cube the scaling factor of the mass is this nice integer power of the scaling factor $\frac{1}{2}$. In fact, is that exponent the dimension of each shape!

Therefore what it means for a shape to be two-dimensional is that if you scale down its length L by a factor of s $(s \cdot L)$ its mass M will be scaled down by that factor raised to the second power $(s^2 \cdot M)$. Analogously for three-dimensional shapes $(s^3 \cdot M)$. If we now apply the same idea to the Sierpinski triangle we know that its dimension should fulfill:

$$\left(\frac{1}{2}\right)^D = \left(\frac{1}{3}\right)$$

Because the Sierpinski triangle is self-similar we know that if we scale the side length by half the mass scales by a third. We can solve this equation to determine the dimension of the Sierpinski triangle:

$$2^D = 3 \Rightarrow D \cdot \log_2 2 = \log_2 3 \Rightarrow D \approx 1.585$$

In general, is it possible to have shapes with any positive real number as a dimension. The fractal dimension of the shapes we determined here is the Hausdroff dimension.
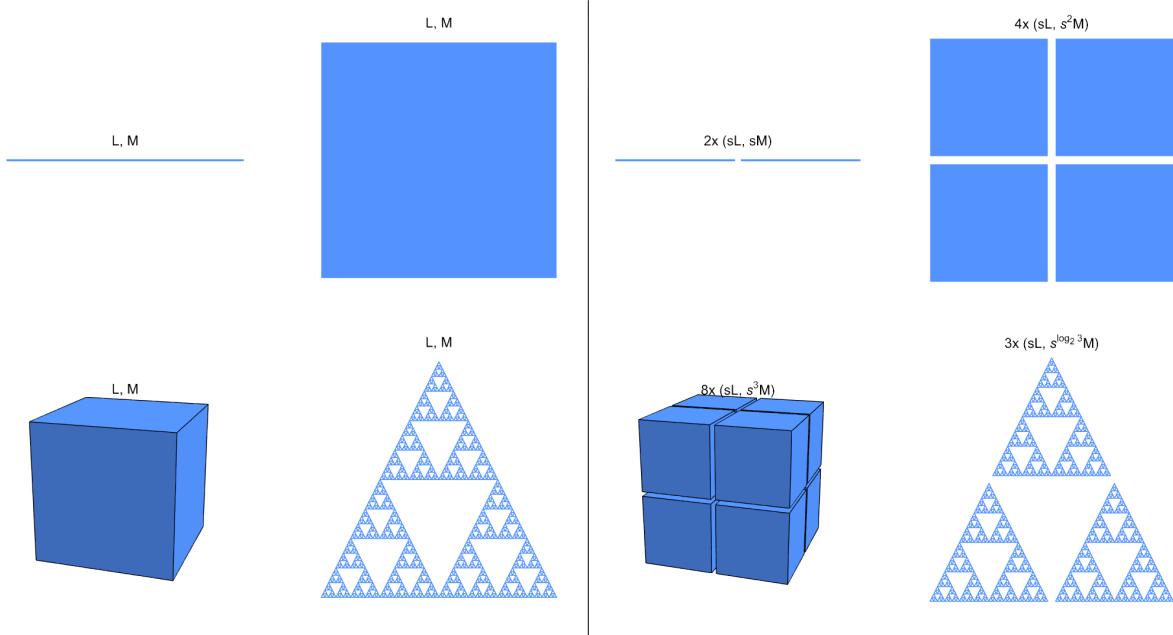

Figure 4: The four shapes and the scaled down versions for $s = \frac{1}{2}$.

2.2.2 Definition of a fractal

Once you have understood what a fractal dimension is the definition of a fractal is quite simple. Mandelbrot defined fractals as "set[s] for which the Hausdorff–Besicovitch dimension strictly exceeds the topological dimension."[1] [5].

For all traditional geometric shapes, the Hausdroff dimension is an integer agreeing with the usual sense of dimension we know, which is formally called the topological dimension[7]. For example, is the topological dimension of the Sierpinski triangle one and the Hausdroff dimension as we saw $\log_2(3)$. The topological dimension of the line, square and cube are equal to the Hausdroff dimension (one, two and three). Therefore are these shapes not fractals. For the Sierpinski triangle does the Hausdroff dimension of ≈ 1.585 strictly exceed its topological dimension of 1 which makes it a fractal.

The Hausdroff dimension has a proper definition that generalizes to all the different shapes but is just to technical too include here.

2.2.3 Identifying fractals

Because the definition of a fractal heavily relies on the fractal dimension the task of identifying fractals and analyzing if a given shape is a fractal mainly reduces to the task of determining the fractal dimension of a given shape.

Because the examples I choose in 2.2.1 were chosen for clarity, and the scaling unit and ratios were already known ahead of time I haven't told you anything about how we determine or rather measure the fractal dimension of arbitrary shapes which are not self-similar like the coastline of Britain as a very famous example. We already know that Mandelbrot used the Hausdroff dimension to define fractals but there are several other formal mathematical definitions of different types of fractal dimensions. Although in some cases all these dimensions coincide, in general, they are not equivalent but often easy to implement algorithmic approximations [6]. The one I want to show you and use to identify fractals is the Minkowski dimension or also known as the box-counting dimension. It coincides with the Hausdroff dimension in many cases and in some sense just counts using boxes instead of balls.

The general idea goes as follows: For a 2D image of a shape, we count how many boxes of different-sized grids touch the shape. This gives us a kind of measure for how details of a shape change with its scale. Note that scaling down the grid (size of the individual squares) is equivalent to scaling up the shape itself. The number of boxes N depends on the scaling factor and will follow some kind of polynomial with the dimension of the shape as the exponent.

$$N(s) = c \cdot s^D$$

Because figuring out the exponent of a polynomial is hard we can instead take the logarithm on both sides. That way the dimension drops down from the exponent.

$$\log N(s) = \log c + D \cdot \log s$$

That means if we plot N over s on a log-log plot the dimension will be the slope of the line.

Additionally, instead of working with the limit $D = \lim_{s \to \infty} \frac{\log N(s)}{\log s}$ we sample values of N for a few different values of s that work for the size of our image and fit a line through them. This gives a good enough approximation.

The land frontier of Germany for example is not self-similar. We can approximate its Hausdroff dimension by measuring its Minkowski dimension. The log-log plot gives us a clean fit which means that the fractal dimension of the image is very constant over many different scales.
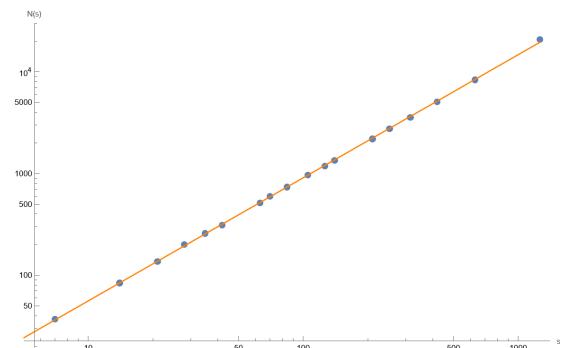

Figure 5: The log-log plot for measuring the Minkowski dimension of the frontier of germany.

For the used image the empirical dimension of the land frontier of Germany turns out to be 1.21 which is close to 1.15 the value Mandelbrot himself calculated. [13].

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[1]Although this turned out to be a bit too restrictive for him in the following years guiding him to simplify the definition a bit.
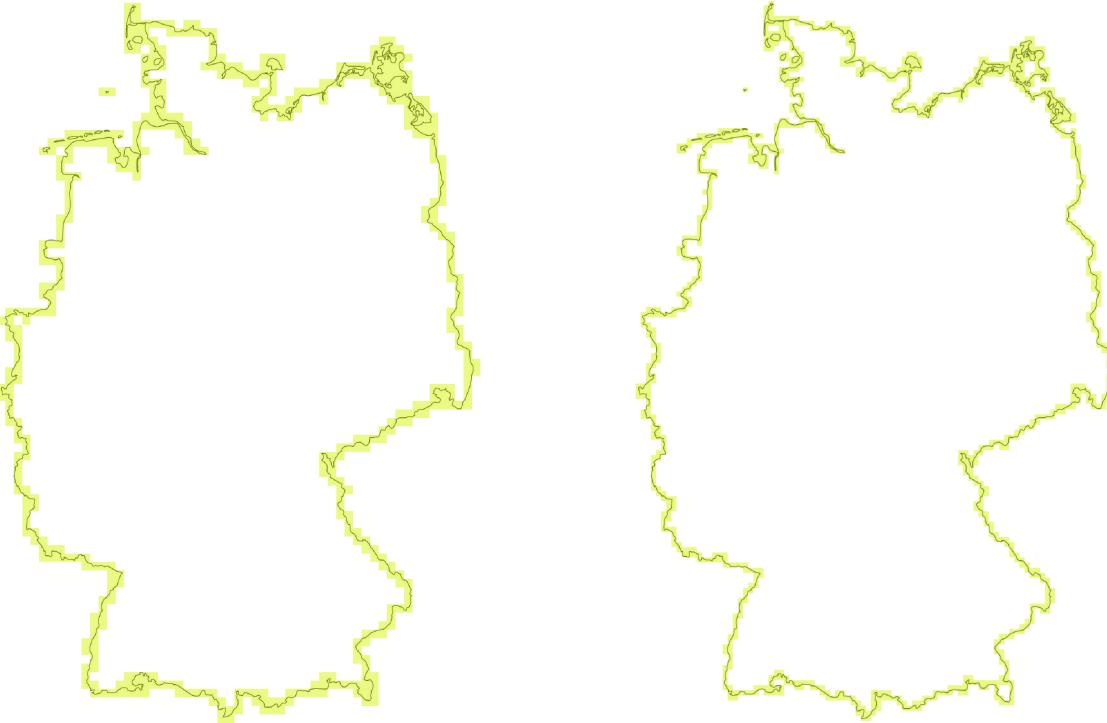
Figure 6: Two examples of the number of boxes touched by the land frontier of Germany for different sized grids.

2.3 Fractals

With a solid foundation on what fractals are I now want to show you four of the most well-known fractals and tell you about the math behind them.

In general, can fractals be categorized by the different techniques for generating them. Apart from the ones which emerge from simple geometric rules which are called Iterated function systems (IFS) like the Sierpinski triangle I will be focusing on the most common ones which are non-trivial: Escape-time fractals and one Strange Attractor [5]. Escape-time fractals are generated by iterating a formula on each point in a given space. If a point diverges as the formula is iterated, it escapes otherwise, it remains bounded. To get an image we color each point based on whether the system remains bound or not and if not how fast it diverges. The majority of relevant Escape-time fractals rely heavily on holomorphic dynamics as you might have already guessed. Strange Attractors are attractors

called that have a fractal structure [2] while attractors are roughly speaking regions that attract points moving around a space according to some fixed law which causes them to stay bounded in that region and sometimes bunch up.

Looking again at the general structure of a dynamic system $z_{k+1} = f(z_k)$ we can see that there are two major things that can be changed as input to have an effect on the behavior of the system. We can change the seed value z_0 which is the first value that gets plugged into our function f or we can change the function itself. From that arise two fundamental ways in which we can visualize such a system. We can have a fixed function and change the seed value based on the coordinate (pixel of our image) or we can fix the seed value and change the function based on the coordinate.

2.3.1 The Mandelbrot set & Julia sets

The Mandelbrot set named after Benoît Mandelbrot and its corresponding Julia sets named after Gaston Julia are by far the most well-known fractals. They are Escape-time fractals and their iterated holomorphic function called the quadratic map looks like this:

$$z_{n+1} = z_n^2 + c, \quad z_n, c \in \mathbb{C}$$

The Mandelbrot set and Julia sets of the quadratic map are in a way different interpretations or visualizations of the same math. For the Mandelbrot set we have one seed value ($z_0 = 0$) and are altering the quadratic map by

changing c based on the coordinate. For a Julia set we are now fixing the function ($c = $ const.) and change the seed value z_0 based on the coordinate. This leads to a Julia set for every $c \in \mathbb{C}$. For both, we are coloring the corresponding point black if the iteration stays bound (either in a cycle, at a fixed point or in a chaotic but bounded sequence) or some other color or gradient if and how fast the system diverges to infinity. What helps a lot with the computation is that mathematicians have shown that for $\|z_n\| > 2$ the system will definitely diverge. This is the iconic Mandelbrot set:
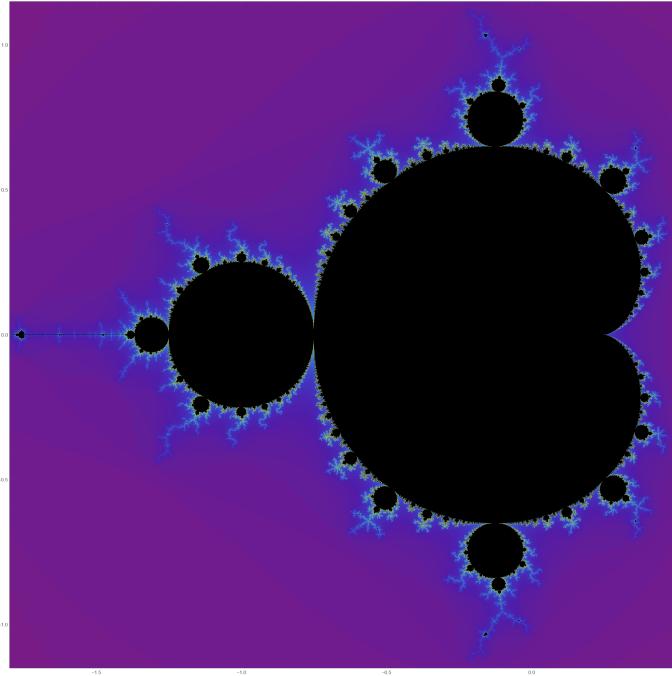
Figure 7: The Mandelbrot set.

Remember: All the black regions are where the system stays bound. To get a feel for it we can look at some iterations.
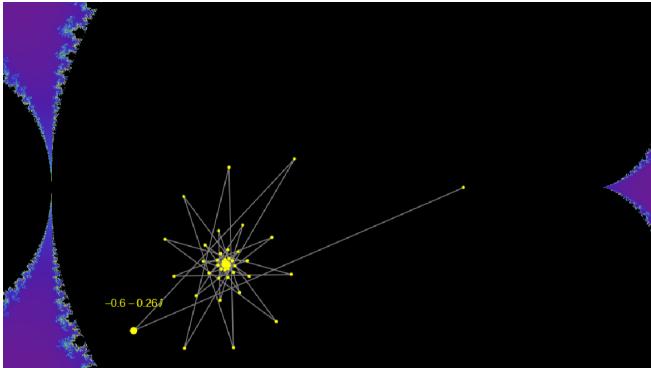

Figure 8: A close-up of the Mandelbrot set with the first 50 values of the quadratic map $(z_1, \ldots, z_{50})$ for $c = -0.6 - 0.26i$.

As we can see for $c = -0.6 - 0.26i$ the quadratic map converges to a fix point.
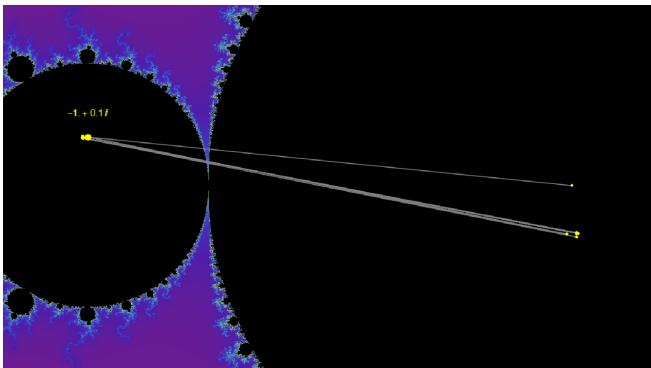

Figure 9: A close-up of the Mandelbrot set with the first 50 values of the quadratic map $(z_1, \ldots, z_{50})$ for $c = -1 + 0.1i$.

For $c = -1 + 0.1i$ the quadratic map results in a two-cycle.

More generally does the quadratic map have an attractive fixed point for all the parameters c in the main cardioid which is the biggest black bulb in the center. The system always converges to a single point in that cardioid. The second biggest circular bulb left to the main cardioid corresponds to parameters for c which all end in a cycle of period two. Likewise, do all the big bulbs correspond to a specific attractive n-cycle. All these interior components of the Mandelbrot set which have an attractive periodic cycle are called hyperbolic components.
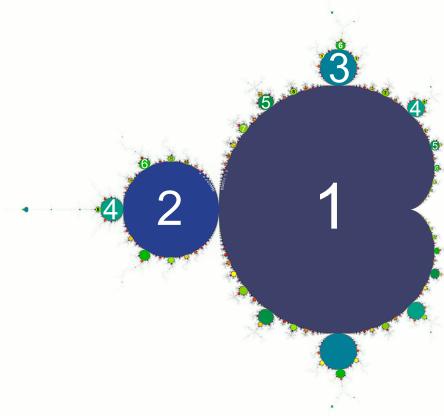
Figure 10: Periods of the hyperbolic components of the Mandelbrot set [12].

The Julia sets work in the same way. Now c is fixed and the resulting behavior of the quadratic map for different values of z_0 is examined.

Although I also sometimes used the term Julia set for referring to the specific image of the fractal we get for a given c one point I want to make clear is that the term Julia set actually has a much broader meaning going way beyond the $z^2 + c$ case. The term also does not refer to the black region of the shape itself as many think but describes

its boundaries.

Somehow something is very fundamental about the Mandelbrot set. Even π and the Fibonacci sequence show up in it [11]. Moreover, the Mandelbrot set itself even shows up in some other fractals.
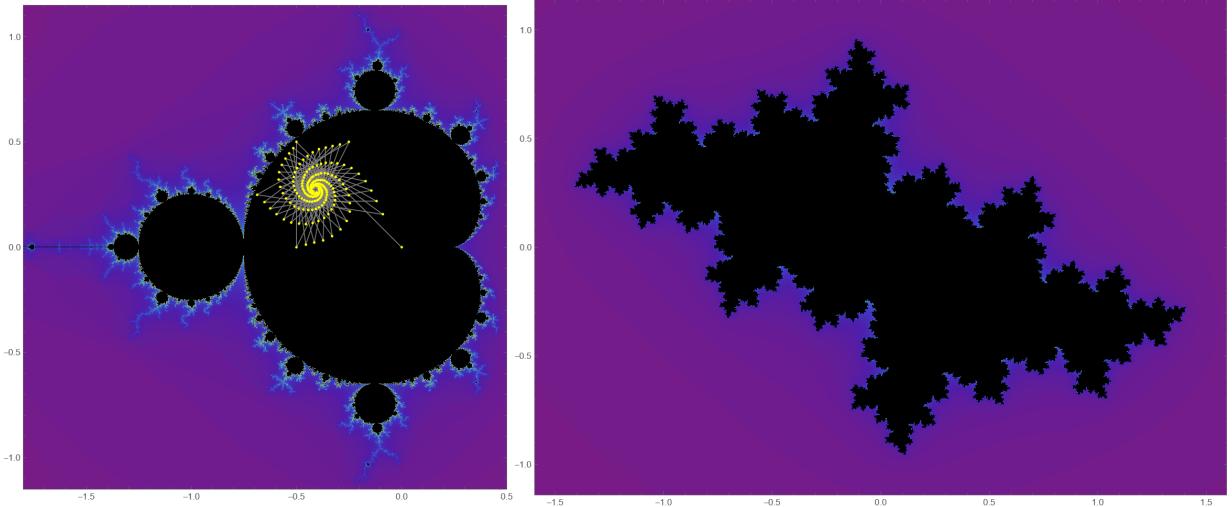

Figure 11: The Mandelbrot with the iteration for $c = -0.5 + 0.5i$ and the corresponding Julia set

2.3.2 Newton fractal

The Newton fractal originates from and is named after Newton's method (also known as Newton-Raphson method) for finding the roots of a holomorphic function f[2]. It is an algorithm where (under a few assumptions) any z_{k+1} is closer to one of the roots of f than its predecessor z_k [14]. The fractal emerges when applying Newton's method to all the points of the complex plane for a given function like $f(z) = z^3 - 1$ and coloring the original coordinate based on the root it landed on. One step with Newton's method looks like this:

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}$$

$f(z) = z^3 - 1$ has three complex roots: 1, $e^{\frac{2\pi}{3}i}$ and $e^{-\frac{2\pi}{3}i}$.
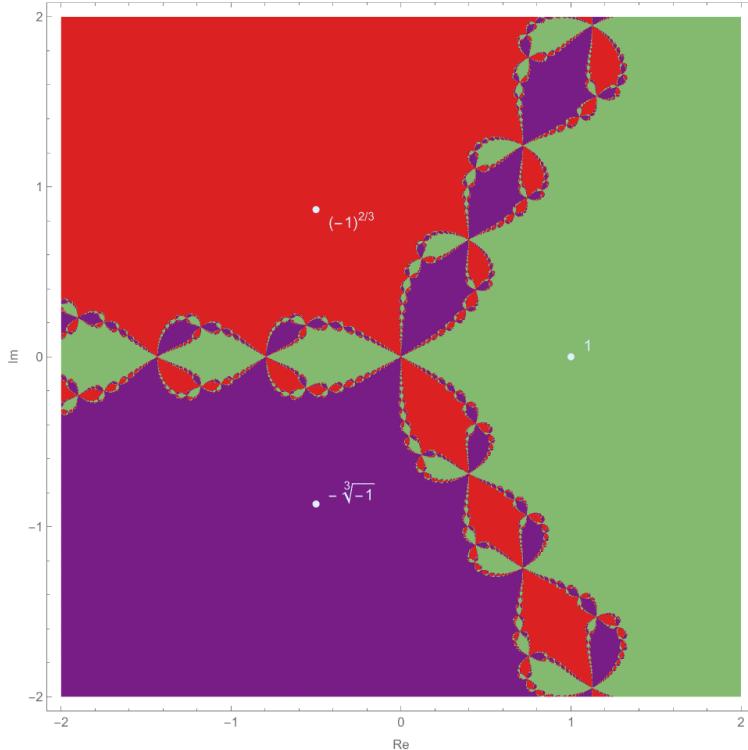
Figure 12: Newton fractal for $f(z) = z^3 - 1$.

[2]Even though the Newton fractal is named after Isaac Newton he knew nothing about this fractal.

2.3.3 The bifurcation diagram of the logistic map

The logistic map is a polynomial function (mapping) of degree 2 and is the archetypal example of how complex and chaotic behavior can arise from simple non-linear dynamical equations [10]. Mathematically the logistic map looks like this:

$$x_{n+1} = r \cdot x_n(1 - x_n), \quad x_n \in [0,1], \ r \in [0,4]$$

Notice that this is not a holomorphic function anymore. x_n and r are both real-valued variables. Additionally, we will only look at values of r in the range of 0 to 4 where x_n will be bounded between 0 and 1.

Like with all the other fractals so far the logistic map also forms a sequence. Instead of visualizing the different behaviors that can arise (Mandelbrot/Julia set: bound-/unbound, Newton fractal: converges to point $a_0/a_1/...$) we now visualize the concrete values x_n the system results in for a given input. We already know that for any given $r \in [0,4]$ x_n is bound. The remaining behaviors the system could show are the convergence to a single point, cyclic behavior and chaotic behavior. The plot of the logistic map with r on the x-axis and the values of the state the system approaches on the y-axis is called the bifurcation diagram of the logistic map. The seed value $x_0 \in (0,1)$ has over the many iterations no influence on the eventual state of the system and can therefore be picked randomly. The logistic map is a Strange Attractor and forms attracting cycles for any $x_0 \in [0,1]$.
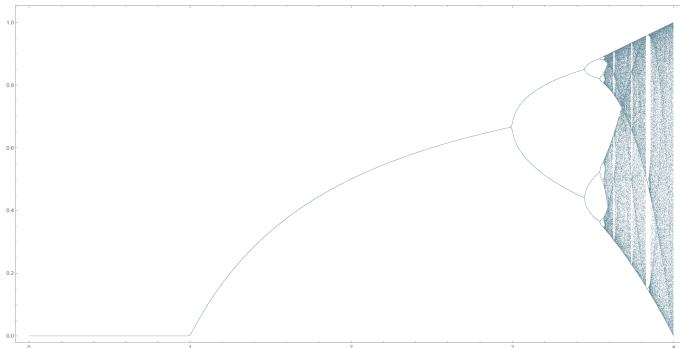
Figure 13: The bifurcation diagram of the logistic map for $r \in [0,4]$.

The bifurcation diagram shows that for $r \in [0,3]$ the system converges to a single point. For values of r below 1 always to 0[3]. Some examples showing x_n change with each iteration for a specific r make this very clear.
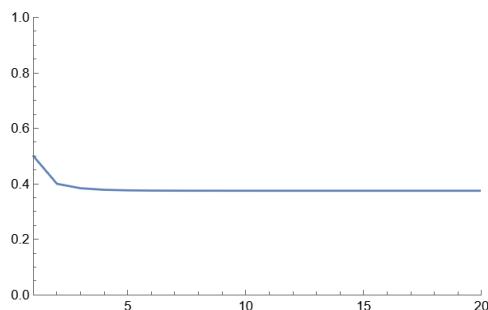
Figure 14: The logistic map converging to 0.375 for $r = 1.6$ with arbitrary $x_0 = 0.5$.

For $3 < r \leq\approx 3.45$ the system always ends in a two-cycle creating the two big branches or in other words the first big bifurcation in the bifurcation diagram.
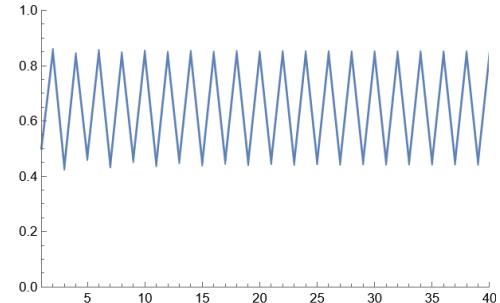

Figure 15: The logistic map resulting in a two-cycle (0.45, 0.84) for $r = 3.4, x_0 = 0.5$.

Going even further in the interval roughly from 3.45 to 3.55 after a few iterations the values stay bound in a four-cycle.
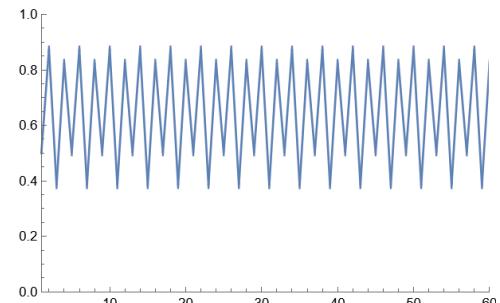

Figure 16: The logistic map resulting in a four-cycle (0.5, 0.88, 0.38, 0.82) for $r = 3.5, x_0 = 0.5$.

After that chaos with a few regions of order with n-cycles emerge.
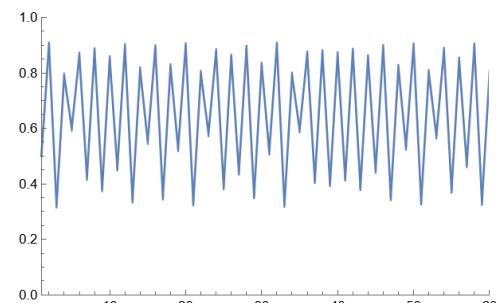
Figure 17: The logistic map resulting in chaos for $r = 3.6, x_0 = 0.5$.

The bifurcation diagram of the logistic map shows self-similarity. For all of the non-chaotic points for example at the three arms around $r \approx 3.82$ the diagram contains a shrunk and slightly distorted version of the whole diagram. The bifurcation diagram is an example of the deep and ubiquitous connection between chaos and fractals [10].

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[3]Originally the logistic map and its bifurcation diagram come from biology and were used to model the population of species with r being the growth rate. The $r < 1$ case refers to the case where the growth rate is so low that a given population goes extinct.
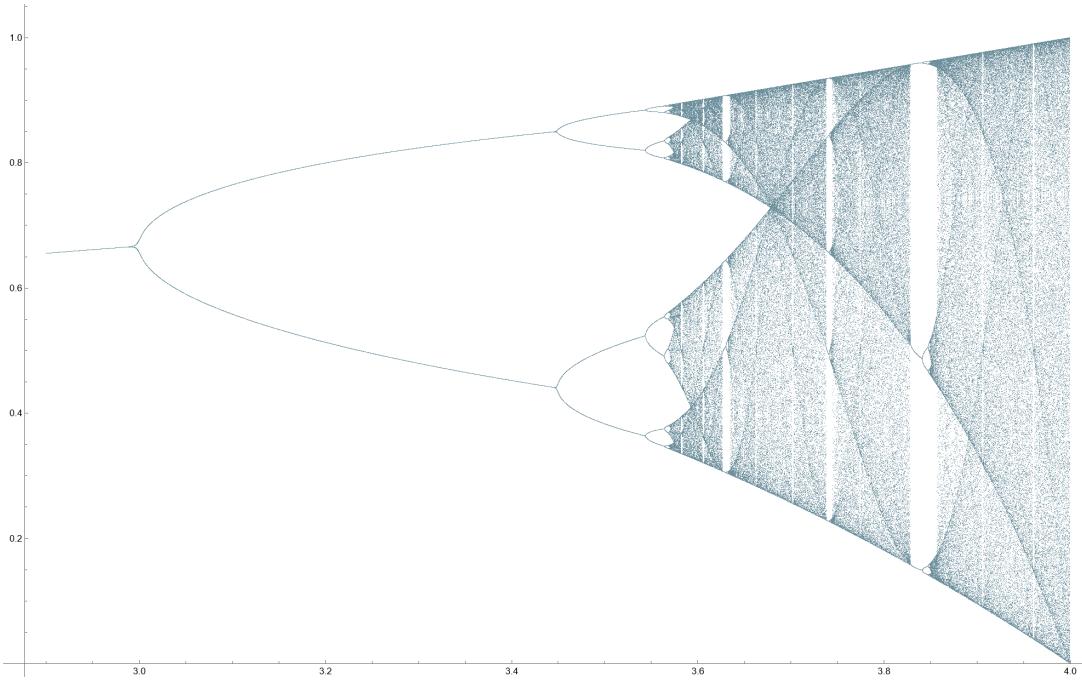
Figure 18: The bifurcation diagram of the logistic map for $r \in [2.9, 4]$.

2.4 The logistic map in the Mandelbrot set

Similar to how the actual iterated values of the logistic map are directly plotted to create the bifurcation diagram we can extend the plot of the Mandelbrot set into a third dimension to include the data about the values z_n takes eventually after many iterations. Because we work with complex functions which we can only visualize in a plane we have to reduce the values to a real-valued number that we can visualize in a single dimension. In my demonstration I simply reduce the complex numbers to their real part ($\mathrm{Re}(c \in \mathbb{C})$).
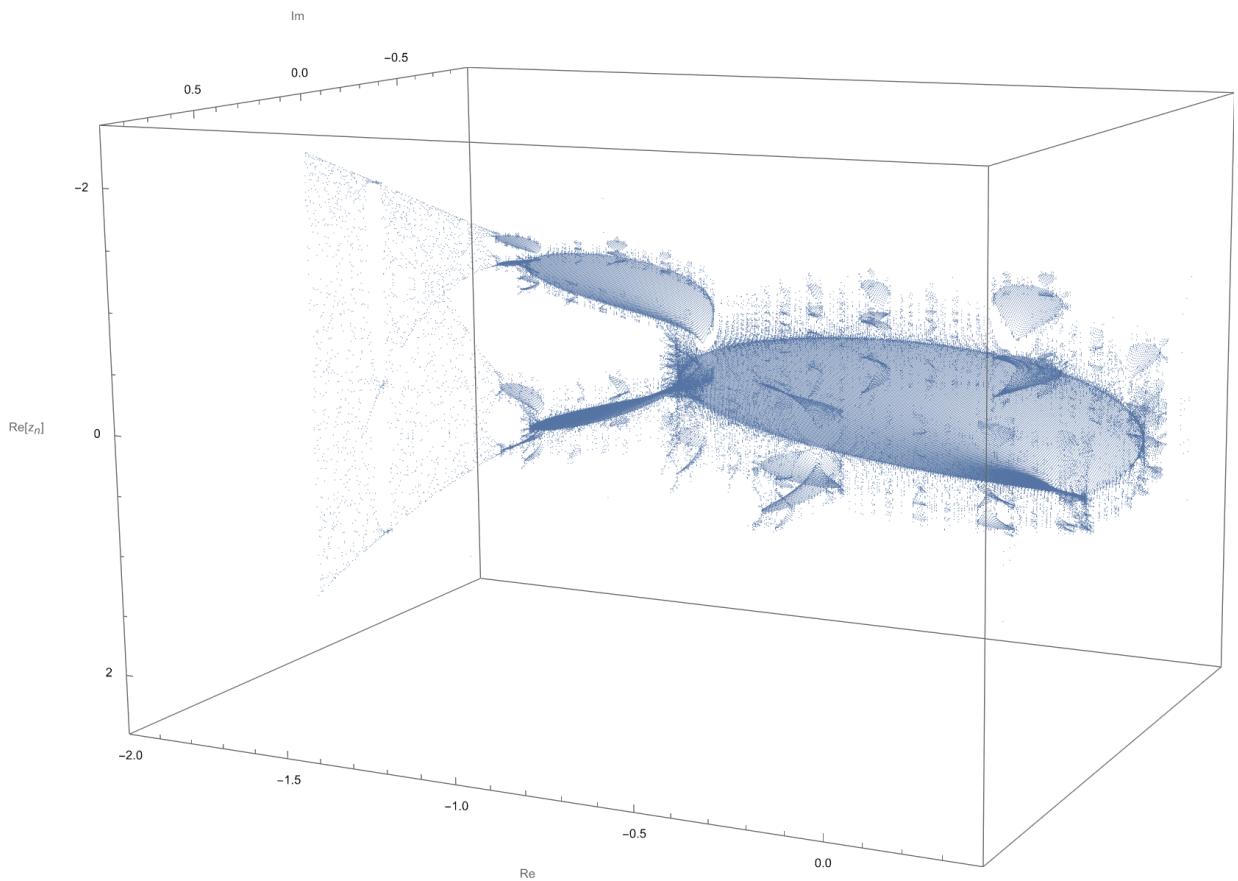
Figure 19: The Mandelbrot set in 3D.

The view from the top (Figure 20) shows the normal image of the Mandelbrot set we are used to.
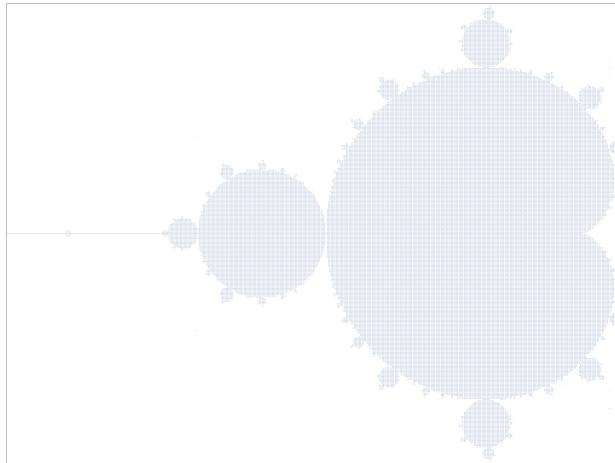
Figure 20: The Mandelbrot set in 3D - top-down view.

The intersection of the Mandelbrot set with the real axis is precisely the interval $\left[-2, \frac{1}{4}\right]$. If we look exactly at that slice (Figure 21) we are able to make a fascinating observation. A linearly transformed version of the bifurcation diagram of the logistic map is part of the Mandelbrot set!
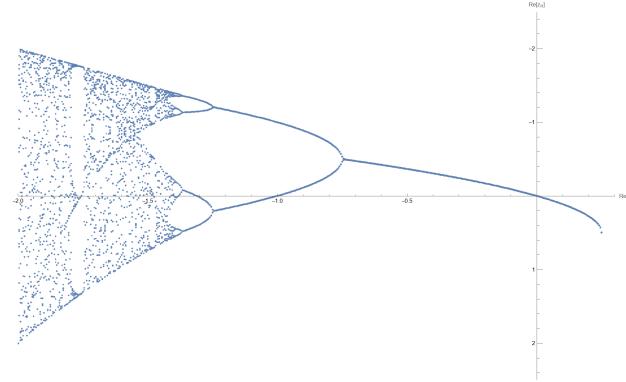
Figure 21: The Mandelbrot set in 3D - slice at Im = 0.

3   Conclusion

Working with fractals and visualizing them mostly comes down to brute force calculations but they are truly beautiful and in some way philosophical shapes. Simple rules can create phenomenal complexity which we humans could never draw with a pen and a piece of paper. What makes them even more fascinating to me is the fact that fractals come from a time when mathematicians did not even have the computational power to visualize what we are seeing today and what I showed you in this small article. Additionally do fractals not just arise naturally in nature but they are so common that a fractional fractal dimension seems to be the core differentiator between objects that arise naturally and those that are human-made. Like Mandelbrot said "beautiful, damn hard, increasingly useful. That's fractals."[5].

There is much more to learn and discover out there I could get a small glimpse at but just could not cover in this short article. Like the title says this is meant as an introduction. It is meant to inspire and encourage you to explore. I want you to ask yourself: "Why do fractals even occur in the first place?". Let me tell you: It is not just because math is funky sometimes. Mathematicians have actually built up a solid understanding of what they are and why they emerge. I hope I could spark your interest to browse the internet and dive deep into the wonderful field of fractal geometry. Lastly I want to emphasize how much this article was build upon the videos from Grant Sanderson from 3Blue1Brown (you can find his videos here [16] [15] [17]).

Finally two more beautiful fractals:
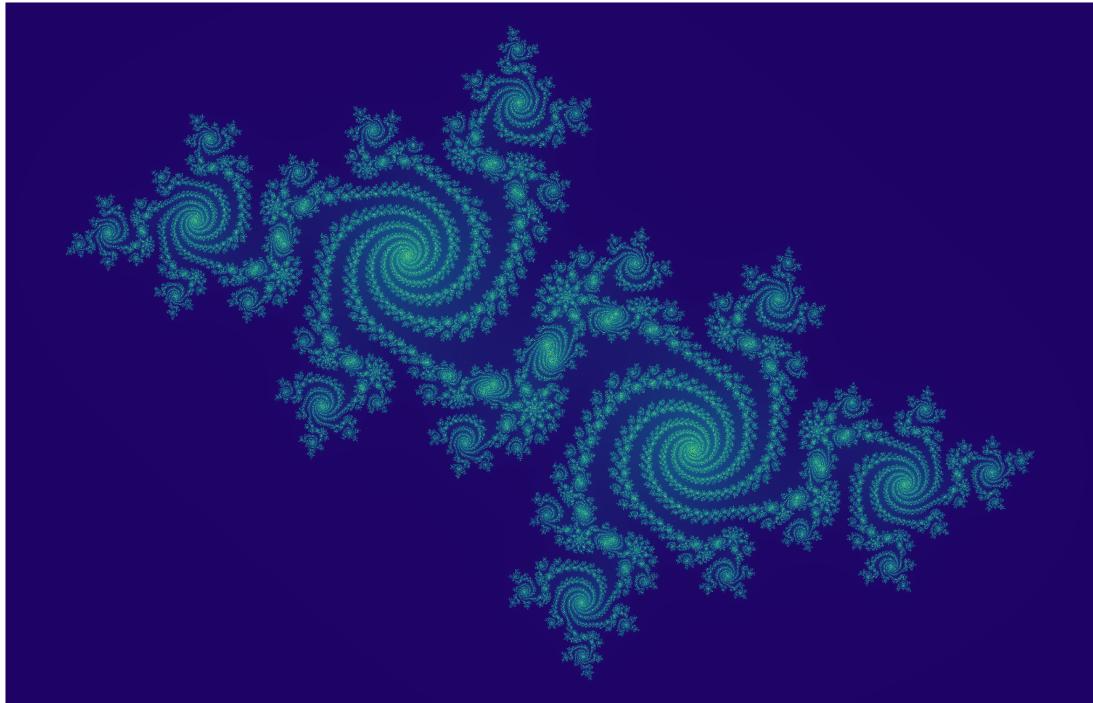
Figure 22: Julia set of the quadratic map for $c = -0.512511498387847167 + 0.521295573094847167i$.
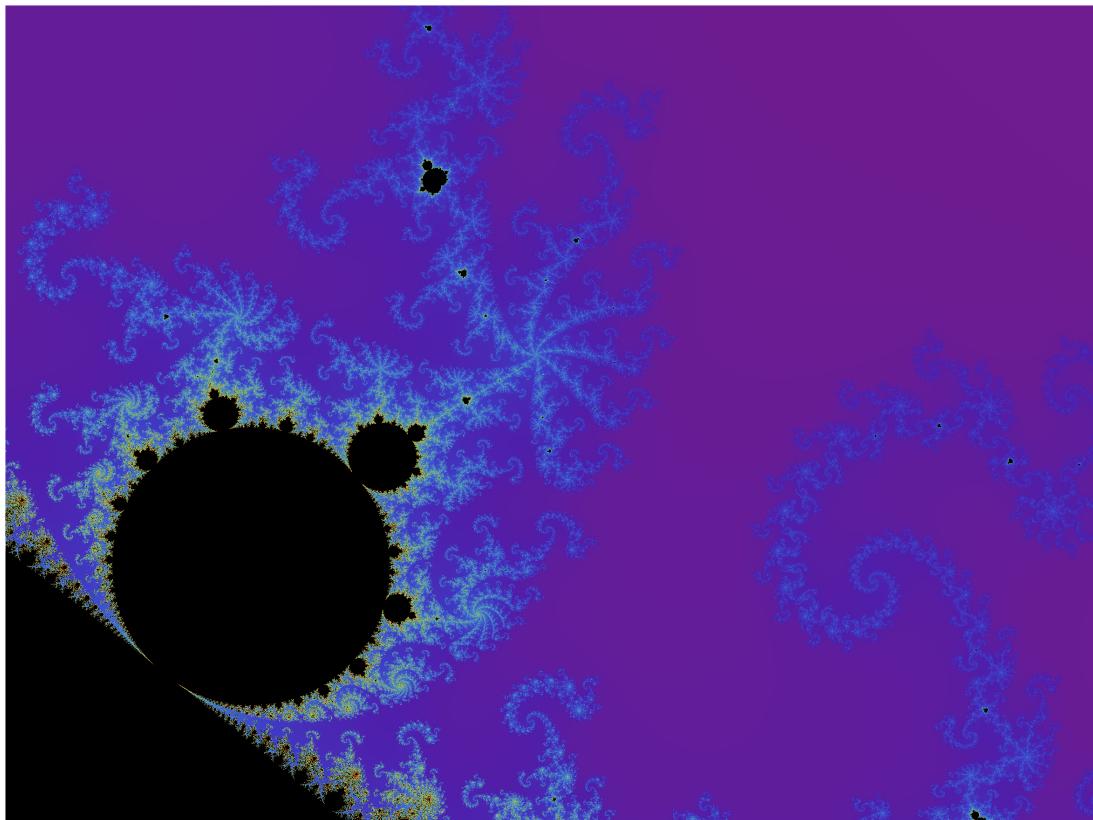

Figure 23: Close-up of Mandelbrot set around $-0.836 + 0.229i$.

Mathematica Code

The code for all the visuals written in Wolfram Mathematica 13 can be found on my website: murphy-in.space.

Bibliography

[1] A History of Fractal Geometry. MacTutor. URL: https://mathshistory.st-andrews.ac.uk/HistTopics/fractals/.

[2] Attractors. Wikipedia. URL: https://en.wikipedia.org/wiki/Attractor#Strange_attractor.

[3] Benoit Mandelbrot. Wikipedia. URL: https://en.wikipedia.org/wiki/Benoit_Mandelbrot.

[4] Complex dynamics. Wikipedia. URL: https://en.wikipedia.org/wiki/Complex_dynamics.

[5] Fractal. Wikipedia. URL: https://en.wikipedia.org/wiki/Fractal.

[6] Fractal dimension. Wikipedia. URL: https://en.wikipedia.org/wiki/Fractal_dimension.

[7] Hausdorff dimension. Wikipedia. URL: https://en.wikipedia.org/wiki/Hausdorff_dimension.

[8] Holomorphic function. Wikipedia. URL: https://en.wikipedia.org/wiki/Holomorphic_function.

[9] Ivar Leidus. Romanesco broccoli, showing self-similar form approximating a natural fractal. URL: https://commons.wikimedia.org/w/index.php?curid=100133434.

[10] Logistic Map. Wikipedia. URL: https://en.wikipedia.org/wiki/Logistic_map.

[11] Mandelbrot set. Wikipedia. URL: https://en.wikipedia.org/wiki/Mandelbrot_set.

[12] Mandelbrot Set – Periodicities coloured. URL: https://upload.wikimedia.org/wikipedia/commons/0/0e/Mandelbrot_Set_%E2%80%93_Periodicities_coloured.png.

[13] Benoît Mandelbrot. How long is the coast of Britain? - Statistical self-similarity and fractional dimension. URL: https://users.math.yale.edu/~bbm3/web_pdfs/howLongIsTheCoastOfBritain.pdf.

[14] Newton's method. Wikipedia. URL: https://en.wikipedia.org/wiki/Newton%5C%27s_method.

[15] Grant Sanderson. Beyond the Mandelbrot set, an intro to holomorphic dynamics. 3Blue1Brown. URL: https://www.youtube.com/watch?v=LqbZpur38nw.

[16] Grant Sanderson. Fractals are typically not self-similar. 3Blue1Brown. URL: https://www.youtube.com/watch?v=gB9n2gHsHN4.

[17] Grant Sanderson. Newton's method produces this fractal, why don't we teach it in calculus classes? 3Blue1Brown. URL: https://www.youtube.com/watch?v=-RdOwhmqP5s.

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